Salud, Ciencia y Tecnología - Serie de Conferencias. 2023; 2:423
doi: 10.56294/sctconf2023423

SERIE DE CONFERENCIAS

Categoría: Congreso Científico de la Fundación Salud, Ciencia y Tecnología 2022

ORIGINAL

On totient graph of a set of positive integers

Sobre el grafo totiente de un conjunto de enteros positivos

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Citar como: Mishra A, Chandra Dey P. On totient graph of a set of positive integers. Salud, Ciencia y Tecnología - Serie de Conferencias 2023; 2:423. https://doi.org/10.56294/sctconf2023423

Recibido: 12-06-2022 **Revisado:** 31-08-2022 **Aceptado:** 24-10-2022 **Publicado:** 25-10-2022

ABSTRACT

In this paper, we introduce a new class of graphs called *totient graph* of the set of the first n positive integers. A totient graph, denoted by $T(I_n)$, is a simple, undirected graph with vertex set $I_n = \{1,2,...,n\}$ and any two distinct vertices x and y are adjacent if and only if $x + y \in P(I_n)$, where $P(I_n) = \{x \in I_n : x \equiv6 0(\text{mod}\varphi(n))\}$.

Keywords: Degree; Eulerian; Euler's φ−function; Connected Graph.

RESUMEN

En este trabajo introducimos una nueva clase de grafos denominada grafo totiente del conjunto de los n primeros enteros positivos. Un grafo totiente, denotado por T(In), es un grafo simple, no dirigido, con un conjunto de vértices $I_n = \{1,2,...,n\}$y dos vértices cualesquiera x e y son adyacentes si y sólo si $x + y \in P(I_n)$, donde $P(I_n) = \{x \in I_n : x \equiv6 0(\text{mod}\varphi(n))\}$.

Palabras clave: Grado; Euleriano; Función φ de Euler; Grafo Conectado.

INTRODUCTION

Before delving into the main findings of the article, we begin with some preliminary definitions and concepts involved in graph theory.

Definition 1.1:[(12)] Let G be a (p,q) graph. Then G is said to be connected if for each pair of vertices, there exists at least one path which joins them.

Definition 1.2:[(12)] A graph G is said to be a null graph if it contains no edges.

Definition 1.3:[(12)] A graph G is said to be Eulerian if it can be traversed without retracing any of its edges.

Definition 1.4:[(12)] A graph G is said to be Hamiltonian if it can be traversed without visiting any of its vertices more than once.

Definition 1.5:[16] For any graph G, the diameter of G, denoted by $diam(G)$ is given by $diam(G)$ = $sup\{d(x,y)$: where x and y are distinct vertices of G$\}$ and $d(x,y)$ is the length of the shortest path joining x and y.

Definition 1.6:[16] The *girth* of the graph G, denoted by $gr(G)$, is the length of the shortest cycle in G. If G contains no cycles, then $gr(G) = \infty$.

Definition 1.7:[16] A non-empty subset S of the set of all the vertices V of a graph is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S.

Definition 1.8:[16] The *domination number* γ of a graph G is defined to be the minimum cardinality of a dominating set in G and the corresponding dominating set is called a γ−set of G.

Definition 1.9:[16] A non-empty subset S of the vertices of a graph G is said to be *independent* if no two vertices in S are adjacent.

Definition 1.10:[16] The maximum cardinality of an independent set of a graph G is called the *independence number* of the graph G and is denoted by $\beta_0(G)$.

Definition 1.11:[16] A graph G is called *excellent* if for every vertex v of G, there exist a γ−set containing v.

Definition 1.12: The totient $\varphi(n)$ of a positive integer n greater than 1 is defined to be the number of positive integers less than n that are relatively prime to n.

In this article, we introduce a new class of graphs called totient graph of the set I_n of the first n positive integers. Denoted by $T(I_n)$, a totient graph is a simple, undirected graph with vertex set $I_n = \{1,2,...,n\}$ and any two distinct vertices x and y are adjacent if and only if $x + y \in P(I_n)$, where $P(I_n) = \{x \in I_n : x$ ≡6 $0(\mathrm{mod}\varphi(n))\}$. Also, throughout the article, we use the notations $P(I_n)$ to denote the set $P(I_n) = \{x \in \mathrm{N} : x \equiv 0(\mathrm{mod}\varphi(n))\}$ and $V(T(I_n))$ to denote the vertex set of $T(I_n)$.

Example 1.1: Let us consider the set $I_{10} = \{1,2,3,4,5,6,7,8,9,10\}$. Then $\varphi(10) = 4$.

Let $P(I_{10}) = \{x \in \mathrm{N} : x$ ≡6 $0(mod4\}$. The totient graph of the set I_{10}, denoted by $T(I_{10})$, contains all the elements of I_{10} as vertices any any two distinct vertices x and y of this graph are adjacent if and only if $x + y \in P(I_{10})$. Figure 1 shows the totient graph of I_{10}.

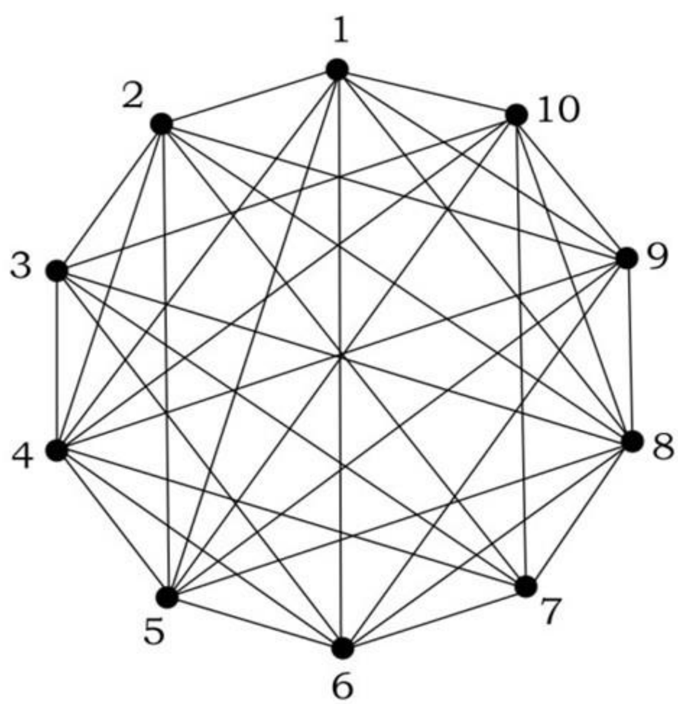

Figure 1. Totient graph on I_{10}

RESULTS

In this section, we characterize some graphical properties of $T(I_n)$.

Theorem 2.1: The graph $T(I_n)$ is connected.

Proof: Let $x \in V(T(I_n))$ such that $x \in P(I_n)$. Then x is adjacent to all the vertices of $I_n \setminus P(I_n)$. Also, the vertex 1 is adjacent to all the vertices of $P(I_n)$ besides all the vertices of the form $\varphi(n) - 1$. Clearly $T(I_n)$ is connected.

Theorem 2.2: Let $n \in \mathbb{N}$ such that $\varphi(n) > 2$. Then for any $i \in V(T(I_n))$,

$$deg(i) = \begin{cases} n - \lfloor \frac{n+i}{\phi(n)} \rfloor + \lfloor \frac{i}{\phi(n)} \rfloor & \text{if } i \in P(I_n) \text{ or if } 2i \in P(I_n) \\ n - \lfloor \frac{n+i}{\phi(n)} \rfloor + \lfloor \frac{i}{\phi(n)} \rfloor - 1 & \text{otherwise} \end{cases}$$

Proof: Let $i \in V(T(I_n))$. Then i is not adjacent to vertices of the form $l.\varphi(n) - i$, $l \in \mathbb{N}$. Let k be the largest positive integer such that $k.\varphi(n) - i \le n$. Then $k \le \frac{n+i}{\phi(n)}$ and $(k+1).\varphi(n) - i > n$ which gives $k+1 > \frac{n+i}{\phi(n)}$. Therefore $k < \frac{n+i}{\phi(n)} < k + 1$. So i is not adjacent to $\lfloor \frac{n+i}{\phi(n)} \rfloor$ vertices of the form $k.\varphi(n) - i$. Again, let l be the largest positive integer such that $l.\varphi(n) - i \le 0$. Then $(l+1).\varphi(n) - i > 0$ and thus $l \le \frac{i}{\phi(n)} < l + 1$. So we don't consider these $\lfloor \frac{i}{\phi(n)} \rfloor$ number of vertices. If $i \in P(I_n)$ or $2i \in P(I_n)$, then i is one of the $\lfloor \frac{n+i}{\phi(n)} \rfloor$ vertices. If $2i \notin P(I_n)$, then i is not adjacent to itself. Summarising all these, we get

$$deg(i) = \begin{cases} n - \lfloor \frac{n+i}{\phi(n)} \rfloor + \lfloor \frac{i}{\phi(n)} \rfloor & \text{if } i \in P(I_n) \text{ or if } 2i \in P(I_n) \\ n - \lfloor \frac{n+i}{\phi(n)} \rfloor + \lfloor \frac{i}{\phi(n)} \rfloor - 1 & \text{otherwise} \end{cases}$$

Corollary 2.3: Let $n \in \mathbb{N}$ such that $\varphi(n) > 2$. Then $T(I_n)$ is never Eulerian.

Proof: From theorem 2.2, since the degrees of the vertices of $T(I_n)$ are of opposite parities, so the graph is not Eulerian.

Theorem 2.4: Let $n \in \mathbb{N}$ such that $n + 1 \not\equiv 0 \pmod{\varphi(n)}$. Then $T(I_n)$ is Hamiltonian. **Proof:** Let n be even. For any $i \in V(T(I_n))$, where $i = 1,2,\ldots,n-1$, since $i + (i+1) = 2i + 1$, which is odd, so $i + (i+1)$ is not divisible by $\varphi(n)$. Therefore i is adjacent to $i + 1$. Also since $1 + n$ is odd, so 1 is adjacent to n. Thus the graph $T(I_n)$ contains a cycle containing all its vertices and is, therefore, Hamiltonian.

Let n be odd. Again, $i + (i+1)$ is odd $\forall\, i = 1,2,\ldots,n-1$ and so i is adjacent to $i + 1$. Also if $n + 1 \not\equiv 0 \pmod{\varphi(n)}$, then clearly 1 is adjacent to n and so $1 \sim 2 \sim \ldots \sim n \sim 1$ is a cycle containing all the vertices of $T(I_n)$. Hence $T(I_n)$ is Hamiltonian.

Theorem 2.5: Let $n \in \mathbb{N}$ such that $\varphi(n) > 2$. Then

(a) $diam(T(I_n)) = 2$.

(b) $gr(T(I_n)) = 3$.

Proof:

(a) Let $x,y \in V(T(I_n))$. If x is adjacent to y, then $d(x,y) = 1$. Suppose x and y are not adjacent. Then $x+y \in P(I_n)$. If $x,y \in P(I_n)$, then $\exists\, z \in I_n \setminus P(I_n)$ such that $x-z-y$ is a 2-path in $T(I_n)$. Hence $d(x,y) = 2$. Again, if $x,y \in I_n \setminus P(I_n)$ such that $x+y \in P(I_n)$, then $\exists\, z \in P(I_n)$ such that $x-z-y$ is a 2-path in $T(I_n)$. Hence $d(x,y) = 2$. Summarizing these, we find that $diam(T(I_n)) = 2$.

(b) Since the vertices 1, 2 and $\varphi(n)$ form a 3-cycle, so $gr(T(I_n)) = 3$.

Theorem 2.6: Let $n \in \mathbb{N}$ such that $\varphi(n) > 2$. Then

(a) $\gamma(T(I_n)) = 2$.

(b) $\beta_0(T(I_n)) = \max\{2, \lfloor \frac{n}{\phi(n)} \rfloor\}$.

(c) $T(I_n)$ is an excellent graph.

Proof:

(a) The set $\{x,y\}$ of cardinality 2, where $x,y \in V(T(I_n))$ such that $x \in P(I_n)$ and $y \in I_n \setminus P(I_n)$ is a dominating set of minimum cardinality. Therefore $\gamma(T(I_n)) = 2$. (b) Let $\lfloor \frac{n}{\phi(n)} \rfloor = 1$. Then the set $\{x,y\}$, where $x,y \in V(T(I_n))$ such that $x \in I_n \setminus P(I_n)$ and $y = \varphi(n) - x$ is an independent set of maximum cardinality. Let $\lfloor \frac{n}{\phi(n)} \rfloor \geq 2$. Then the set $\{x,y\}$ of cardinality $\lfloor \frac{n}{\phi(n)} \rfloor$, where $x,y \in P(I_n)$, is an independent set of maximum cardinality. Therefore $\beta_0(T(I_n)) = \max\{2, \lfloor \frac{n}{\phi(n)} \rfloor\}$.

(c) From (a), the graph $T(I_n)$ is clearly excellent since each vertex of $T(I_n)$ is a member of at least one γ-set of $T(I_n)$.

Theorem 2.7: Let I_n be such that n is a prime. Then the size of $T(I_n)$ is $\frac{(n-1)^2}{2}$.

Proof: For any prime n, the vertex $n-2$ is not adjacent to 1, itself and to n. Therefore $deg(n-2) = 3$. The vertices $n-1$ and $\frac{n-1}{2}$ are not adjacent to itself. Therefore $deg(n-1) = deg(\frac{n-1}{2}) = n-1$. For any $i \in V(T(I_n))$ such that $i \neq n-2, n-1, \frac{n-1}{2}$, i is not adjacent to itself and to $n-1-i$. Therefore $deg(i) = n-2$.

Clearly, there is one vertex of degree $(n-3)$, two vertices of degree $(n-1)$ and $(n-3)$ vertices of degree $(n-2)$. So the size of $T(I_n) = \frac{(n-3)+2(n-1)+(n-3)(n-2)}{2} = \frac{n^2-2n+1}{2} = \frac{(n-1)^2}{2}$.

Example 2.1 Figure 2 shows the totient graph on the set I_7

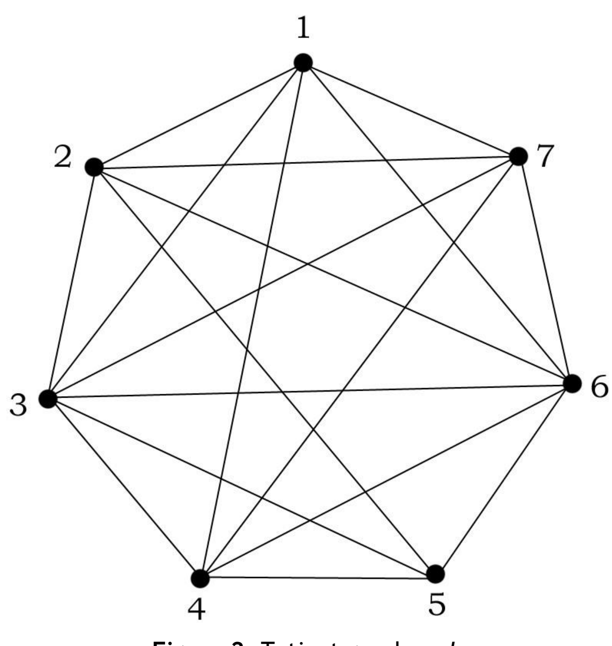

Figure 2. Totient graph on I_7

Theorem 2.8: Let I_n be such that $n = 2^k$ for any $k(\geq 2) \in \mathbb{N}$. Then the size of $T(I_n)$ is 22k−1 − 3.2k−1 + 1.

Proof: Let $i \in V(T(I_n))$. Here we consider two cases:

Case 1: Let i be even.

Subcase 1.1: When $2i \equiv 0 \pmod{2^{k-1}}$.

Then i is not adjacent to itself and to the vertex $2^k - i$. Therefore $deg(i) = 2^k - 2$.

Subcase 1.2: When $2i \not\equiv 0 \pmod{2^{k-1}}$.

Then i is not adjacent to itself and to the vertices $2^{k-1}-i$ and $2^k - i$. Therefore $deg(i) = 2^k - 3$.

Case 2: Let i be odd. Then i is not adjacent to itself and to the vertices $2^{k-1} - i$ and $2^k - i$. Therefore $deg(i) = 2^k - 3$.

Summarizing these cases, we have

$$deg(i) = \begin{cases} 2^k - 3 & \text{if } i \equiv 1 \ (mod\ 2) \text{ or if } 2i \not\equiv 0 \ (mod\ 2^{k-1}) \\ 2^k - 2 & \text{if } 2i \equiv 0 \ (mod\ 2^{k-1}) \end{cases}$$

Clearly, there are 2 vertices of degree 2^k-2, 2^{k-1} vertices of degree (2^k-3) and $(2^{k-1}-2)$ vertices of degree (2^k-3). So the size of the graph $T(I_n)$ is $\frac{2(2^k-2)+2^{k-1}(2^k-3)+(2^{k-1}-1)(2^k-3)}{2}$ = 22k−1 − 3.2k−1 + 1.

Example 2.2: Figure 3 shows the totient graph on the set I_8.

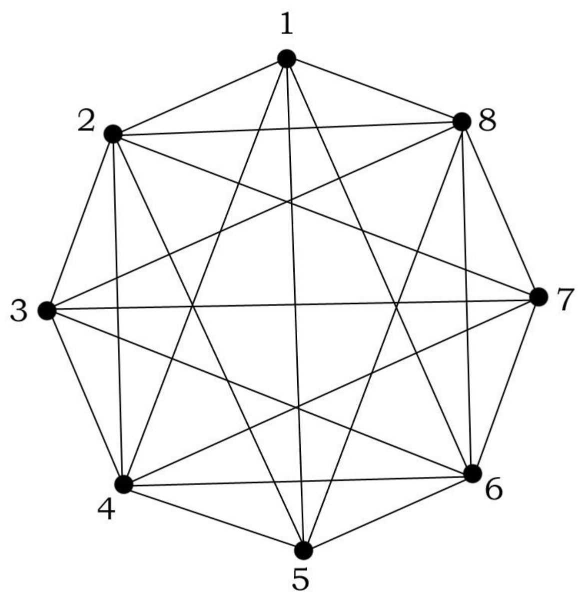

Figure 3. Totient graph on I_8

Theorem 2.9: Let $a_1, a_2 \in I_n$. Then a_1 is adjacent to a_2 in $T(I_n)$ if and only if every element of the coset $a_1 + P(I_n)$ is adjacent to every element of the coset $a_2 + P(I_n)$, provided $a_2 \notin a_1 + P(I_n)$.

Proof: Let a_1 be adjacent to a_2 and $a_2 \notin a_1 + P(I_n)$. Then $a_1 + a_2 \in P(I_n)$ and so $a_2 = z - a_1$, for some $z \in P(I_n)$. The vertices in the coset $a_1 + P(I_n)$ are adjacent to the vertices in the coset $(z - a_1) + P(I_n)$ since for some for $n_1, n_2 \in P(I_n)$, $(a_1 + n_1) + (z - a_1 + n_2) = z + (n_1 + n_2) \in P(I_n)$. Conversely, suppose that each element of the coset $a_1 + P(I_n)$ is adjacent to each element of the coset $a_2 + P(I_n)$. Then for some $n_1, n^1 \in P(I_n)$, $(a_1 + n_1) + (a_2 + n_2) \in P(I_n) \Rightarrow (a_1 + a_2) + (n_1 + n_2) \in P(I_n) \Rightarrow (a_1 + a_2) \in P(I_n)$. Therefore a_1 is adjacent to a_2.

Theorem 2.10: (a) For each $z \in V(T(I_n))$ such that $z \in P(I_n)$ and $2z \in P(I_n)$, the vertices of the coset $z + P(I_n)$ form a null graph.

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[1] $a_1 + (m_1 + m_2) \in P(I_n)$, for some $m_1, m_2 \in P(I_n)$. Likewise, all the elements of the coset $a_2 + P(I_n)$ are adjacent. Therefore $(a_1 + P(I_n)) \cup (a_2 + P(I_n))$ forms a complete

(b) For any $z \in V(T(I_n))$ such that $z \in P(I_n)$ and $2z \in P(I_n)$, the vertices of the coset $z + P(I_n)$ form a complete graph.

(c) Let $a_1,a_2 \in V(T(I_n))$ such that $a_2 \in a_1 + P(I_n)$. Then the graph $(a_1 + P(I_n)) \cup (a_2 + P(I_n))$ is either a null graph or a complete graph.

Proof: (a) Let $z \in P(I_n)$ such that $2z \in P(I_n)$ and let $n_1,n_2 \in P(I_n)$. Then $(z + n_1) + (z + n_2) = 2z + (n_1 + n_2) \in P(I_n) \Rightarrow z + n_1$ is not adjacent to $z + n_2$. Therefore no two elements of the coset $z + P(I_n)$ are adjacent and hence form a null graph.

(b) Let $z \in P(I_n)$ such that $2z \in P(I_n)$ and let $n_1,n_2 \in P(I_n)$. Then $(z+n_1)+(z+n_2) = 2z + (n_1 + n_2) \in P(I_n)$. So all the elements of the coset $z + P(I_n)$ are adjacent and hence form a complete graph.

(c) Since $a_2 \in a_1 + P(I_n)$, let $a_2 = a_1 + z_1$ for some $z_1 \in P(I_n)$. Here two cases arise:

Case 1: Let $2a_1 \in P(I_n)$.

For some $n_1,n_2 \in P(I_n)$, since $(a_1 + n_1) + (a_2 + n_1) = (a_1 + n_1) + (a_1 + z_1 + n_1) = 2a_1 +$
$(2n_1+z_1) \in P(I_n)$, so no two elements of the cosets $a_1+P(I_n)$ and $a_2+P(I_n)$ are adjacent.

Again, no two elements of the coset $a_1 + P(I_n)$ are adjacent since $(a_1 + n_1)+(a_1 + n_2) = 2a_1 + (n_1 + n_2) \in P(I_n)$, for some $n_1,n_2 \in P(I_n)$. Similarly no two elements of the coset $a_2+P(I_n)$ are adjacent. Summarising these, we conclude that $(a_1+P(I_n)) \cup (a_2+P(I_n))$ forms a null graph.

Case 2: Let $2a_1 \in P(I_n)$.

For any $m_1,m_2 \in P(I_n)$, since $(a_1+m_1)+(a_2+m_1) = (a_1+m_1)+(a_1+z_1+m_1) = 2a_1+(2m_1 +z_1) \in P(I_n)$, so the elements of the cosets $a_1 + P(I_n)$ and $a_2 + P(I_n)$ are adjacent.Again, all the elements of the coset $a_1+P(I_n)$ are adjacent since $(a_1+m_1)+(a_1+m_2)$ = graph.

CONCLUSION

With the view to contributing towards the existing literature of number theoretic graph theory, this article introduces a new class of graphs called *totient graphs*. The article also contains detailed and exhaustive examples to bring home the concept to the readers. The authors are hopeful that this paper would open ample scope for further research on the graph defined.

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FUNDING

This research received no external funding.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

AUTHORSHIP CONTRIBUTION

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